

# NUMERICAL STUDY OF THE BEHAVIOR OF A UNIFORM "SPOT" IN AN IDEAL DENSITY-STRATIFIED LIQUID

B. G. Kuznetsov and G. G. Chernykh

UDC 532.5

A numerical analysis of the flow arising as a result of the collapse of a uniform spot in a heavy ideal liquid linearly stratified with respect to density is presented. The nonlinear model of the phenomenon is employed. The numerical results are compared with experiment.

1. We shall consider a plane transient flow arising as the results of the collapse of a uniform "spot" (a region of liquid with constant density) in a heavy ideal liquid stratified with respect to density. We shall assume that the stratification of the liquid is due to the dissolution of salt in it.

The collapse of a spot in a density-stratified liquid and the associated development and propagation of internal waves have been studied in a number of theoretical and experimental investigations. As regards experiments we may cite [1, 2]. The collapse of a spot and the internal waves arising as a result of this were studied numerically in [3], the stratification of the liquid being regarded as weak. This enabled the author to use the Boussinesq [4] approximation to linearize the system of equations describing the problem in relation to density.

A numerical and analytical study of the collapse of a spot was also presented in [5]; it was assumed that the pressure at the boundary of the spot was distributed in accordance with a hydrostatic law. An approximate system of differential equations describing the behavior of a spot was derived in [6]; the resultant approximate system of equations constitutes a linearized system of equations of the hydrodynamics of an ideal liquid and may be solved analytically. In addition to an experimental investigation into the behavior of spots, an approximate system of equations was also derived and solved analytically in [7].

A nonuniform spot with a type of stratification differing from the stratification of the region of liquid outside the spot was studied in [8]. In this form of the problem (in contrast to [3]) it was not assumed that the stratification was weak; the diffusion of the salt in the solution was taken into account. The problem was solved numerically using economical difference schemes [9].

2. We shall assume that the spot is placed in a rectangular tank with walls fairly well removed from the spot and impenetrable to the liquid. The upper and lower walls of the tank we shall call  $\Gamma_1, \Gamma_3$ , the right and left-hand vertical walls  $\Gamma_2, \Gamma_4$ . The spot we shall also consider rectangular (at the initial instant of time), with corresponding boundaries  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ ; it will be regarded as placed centrally in the tank.

Let us denote the region of the tank external to the spot at the initial instant of time by  $\Omega_1$ , and the region of the tank occupied by the spot at the initial instant of time by  $\Omega_2$ , so that the tank constitutes a region  $\Omega_3 = \Omega_1 \cup \Omega_2$ . Now let  $u, v$  be the vector components of the mass-average velocity of the particles of the pure liquid,  $u_1, v_1$  be the vector components of the mass-average velocity of the particles of dissolved salt,  $u_2, v_2$  be the velocity components of the particles of solution,  $\rho$  be the density of the solution,  $\rho = \rho_1 + \rho_2$ , where  $\rho_2 = \text{const}$  is the density of the liquid,  $\rho_1$  is the density of the dissolved salt,  $p$  is the pressure,  $g$  is the gravitational acceleration,  $\lambda = \text{const}$  is the diffusion coefficient of the salt in the solution. The conservation equations of the mass of liquid and the salt dissolved in it and the equations representing the change in the momentum of the mixture take the following form in this nomenclature [10]:

---

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 120-126, May-June, 1973. Original article submitted December 6, 1972.

© 1975 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for \$15.00.

$$\begin{aligned}
u_x + v_y &= 0 \\
\rho_{1t} + (\rho_1 u_1)_x + (\rho_1 v_1)_y &= 0 \\
\rho (u_{2t} + u_2 u_{2x} + v_2 u_{2y}) + p_x &= 0 \\
\rho (v_{2t} + u_2 v_{2x} + v_2 v_{2y}) + p_y &= -g\rho
\end{aligned} \tag{2.1}$$

The lower indices  $x, y, t$  denote differentiation with respect to the corresponding variables.

Adding the relations

$$u_1 - u = -\lambda \rho_1^{-1} \rho_{1x}, \quad v_1 - v = -\lambda \rho_1^{-1} \rho_{1y} \tag{2.2}$$

to the system (2.1) in order to relate the velocity components of the liquid to those of the salt dissolved in it, we obtain a closed set of equations (2.1), (2.2). Making use of Eqs. (2.2) and neglecting terms of the second order in  $\lambda$  in the momentum equations ( $\lambda$  is a small quantity), we obtain the following system of equations describing the problem under consideration:

$$\begin{aligned}
u_x + v_y &= 0 \\
\rho_t + (u\rho)_x + (v\rho)_y &= \lambda (\rho_{xx} + \rho_{yy}) \\
\rho (u_t + uu_x + vu_y) + p_x &= -\lambda \rho_y (v_x - u_y) \\
\rho (v_t + uv_x + vv_y) + p_y &= \lambda \rho_x (v_x - u_y) - g\rho
\end{aligned} \tag{2.3}$$

The problem thus reduces to one of finding the functions  $u, v, p, \rho$  from the system of equations (2.3) with initial conditions

$$\begin{aligned}
u = v = 0, \quad x, y \in \Omega_3, \quad t = 0 \\
\rho = \rho_0 (1 - a\rho_0^{-1} y), \quad x, y \in \Omega_1, \quad t = 0 \\
\rho = \rho_0, \quad x, y \in \Omega_2, \quad t = 0
\end{aligned} \tag{2.4}$$

where  $a = \rho_y$  for  $t=0$ ,  $x, y \in \Omega_1$ ,  $\rho_0 = \text{const}$  (the origin of coordinates is placed in the center of the tank). The unknown functions satisfy the following boundary conditions:

$$\begin{aligned}
v = 0, \quad x, y \in \Gamma_1, \Gamma_3; \quad u = 0, \quad x, y \in \Gamma_2, \Gamma_4 \\
\rho = \rho_0 (1 - {}^{1/2} a \rho_0^{-1} L_y), \quad x, y \in \Gamma_1 \\
\rho = \rho_0 (1 + {}^{1/2} a \rho_0^{-1} L_y), \quad x, y \in \Gamma_3; \quad \rho_x = 0, \quad x, y \in \Gamma_2, \Gamma_4
\end{aligned} \tag{2.5}$$

where  $L_y$  is the vertical dimension of the tank.

Let  $L$  be the horizontal dimension of the tank,  $l_x, l_y$  the horizontal and vertical dimensions of the spot at the initial instant of time.

Introducing dimensionless independent variables and functions

$$\begin{aligned}
x_1 = x/L, \quad y_1 = y/L, \quad \rho_3 = \rho/\rho_0 \\
t_1 = t(ag/\rho_0)^{1/2}, \quad u_3 = u/L(ag/\rho_0)^{-1/2} \\
v_3 = v/L(ag/\rho_0)^{-1/2}, \quad p_1 = pL^{-2}(ag)^{-1},
\end{aligned}$$

transforming to these in the system (2.3), and returning to the original nomenclature, we may write the system under consideration thus:

$$\begin{aligned}
\rho (u_t + uu_x + vu_y) + p_x &= -A\rho_y (v_x - u_y) \\
\rho (v_t + uv_x + vv_y) + p_y &= A\rho_x (v_x - u_y) - B\rho \\
\rho_t + (u\rho)_x + (v\rho)_y &= A(\rho_{xx} + \rho_{yy}) \\
u_x + v_y &= 0
\end{aligned} \tag{2.6}$$

where

$$A = \lambda L^{-2} (ag/\rho_0)^{-1/2}, \quad B = \rho_0 (aL)^{-1}$$

Transforming to dimensionless quantities in the initial conditions (2.4), we rewrite these in the form

$$\begin{aligned}
u = v = 0, \quad x, y \in \Omega_3, \quad t = 0 \\
\rho = 1 - B^{-1} y, \quad x, y \in \Omega_1, \quad t = 0 \\
\rho = 1, \quad x, y \in \Omega_2, \quad t = 0
\end{aligned}$$

In conditions (2.5), only the value of the dimensionless density of the mixture on  $\Gamma_1, \Gamma_3$  vary, that is,

$$\begin{aligned}\rho &= 1 - (a/2) \rho_0^{-1} L_y, & x, y \in \Gamma_1 \\ \rho &= 1 + (a/2) \rho_0^{-1} L_y, & x, y \in \Gamma_3\end{aligned}\quad (2.7)$$

The dimensionless parameters characterizing the problem will be as follows:  $L_y/L, l_x/L, l_y/L, A, B$ .

3. For the numerical integration of the problem in hand, we introduce the current function  $\psi: u = \psi_y, v = -\psi_x$ . The system (2.6) may then be rewritten:

$$\begin{aligned}(\rho\psi_{tx})_x + (\rho\psi_{ty})_y &= f \\ \rho_t + (\Psi_y\rho)_x - (\Psi_x\rho)_y &= A(\rho_{xx} + \rho_{yy})\end{aligned}\quad (3.1)$$

$$\begin{aligned}(f \equiv &\rho(\Psi_x\omega_y - \Psi_y\omega_x) + \rho_x(B + A\omega_x + \Psi_x\Psi_{xy} - \Psi_y\Psi_{xx}) + \rho_y(\Psi_x\Psi_{yy} - \Psi_y\Psi_{xy}) + \\ &+ A\omega_y) + A\omega(\rho_{xx} + \rho_{yy}), \omega \equiv \Psi_{xx} + \Psi_{yy})\end{aligned}$$

The initial conditions will take the form

$$\begin{aligned}\psi &= 0, & x, y \in \Omega_3, t = 0 \\ \rho &= 1 - B^{-1}y, & x, y \in \Omega_1, t = 0 \\ \rho &= 1, & x, y \in \Omega_2, t = 0\end{aligned}\quad (3.2)$$

The boundary conditions (2.5), allowing for (2.7), become

$$\begin{aligned}\psi &= \psi_t \equiv 0, & x, y \in \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \\ \rho &= 1 - (a/2) \rho_0^{-1} L_y, & x, y \in \Gamma_1 \\ \rho &= 1 + (a/2) \rho_0^{-1} L_y, & x, y \in \Gamma_3; \rho_x = 0, x, y \in \Gamma_2, \Gamma_4\end{aligned}\quad (3.3)$$

The vorticity  $\omega$  used here is introduced in order to simplify realization of the numerical algorithm for the solution of the first equation of system (3.1). We require to determine the vorticity on the lines  $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ . Making use of the fact that on  $\Gamma_2, \Gamma_4 \omega = \psi_{xx}$  and allowing for condition (3.3), with the assumption that the first equation of system (3.1) is satisfied on these parts of the boundary, for the vorticity on the corresponding parts of the boundary we obtain

$$\omega_t - (\Psi_x + A\rho^{-1}\rho_y) \omega_y - A\rho^{-1}(\rho_{xx} + \rho_{yy})\omega = 0\quad (3.4)$$

Since at the initial instant of time  $\omega = 0$  along  $\Gamma_2, \Gamma_4$ , it follows from Eq. (3.4) and the first of conditions (3.3) that  $\omega \equiv 0$  on  $\Gamma_2, \Gamma_4$ . On  $\Gamma_1, \Gamma_3 \omega$  may be found by extrapolating with respect to the known values of  $\omega$  inside the tank. It follows directly from the system (3.1) and the conditions (3.2) and (3.3) that the problem is symmetrical with respect to the axis of ordinates. Along the symmetry axis we put  $\omega = \rho_x = \psi = \psi_t = 0$ .

4. The algorithm for the solution of the problem is as follows: for  $t = n\Delta t$ , where  $\Delta t$  is the time step,  $n$  is the number of step in time ( $n \geq 0$ ), the values of  $\rho, \psi$  are known; we find  $\omega = \psi_{xx} + \psi_{yy}$  inside the region  $\Omega_3$  and extrapolate to the upper and lower wall of the tank, then find the right-hand side  $f$  of the first equation of system (3.1), after which we solve the first equation of system (3.1) and find the values of the function  $\psi_t(n\Delta t, x, y)$ ; from the resultant values of  $\psi_t$  we calculate the values of  $\psi[(n+1)\Delta t, x, y]$ . The resultant values of  $\psi$  at the  $(n+1)$ -th step in time are substituted into the second equation of system (3.1), and after solving the latter the process is repeated. This algorithm may easily be realized by means of a finite-difference technique. The mesh is assumed square for the sake of simplicity. The equation  $\omega = \psi_{xx} + \psi_{yy}$  is realized in the following manner [11]:

$$\begin{aligned}\omega_{i,j}^n &= 1/12 h^{-2} (\psi_{i-1,j-1}^n + \psi_{i+1,j+1}^n + \psi_{i-1,j+1}^n + \psi_{i+1,j-1}^n \\ &+ 4(\psi_{i-1,j}^n + \psi_{i,j-1}^n + \psi_{i+1,j}^n + \psi_{i,j+1}^n) - 20\psi_{i,j}^n) \\ \psi_{i+\alpha,j+\beta}^n &\equiv \psi((n\Delta t, (i+\alpha)h, (j+\beta)h))\end{aligned}$$

where  $h$  is the step of the mesh.

In seeking the right-hand side  $f$  of the first of the equations in system (3.1), we use the ordinary central differences. We put  $\psi_t \equiv z$ ; in order to find  $z$  we use the iterative scheme of a stabilizing correction [9] approximating the first equation of system (3.2) which incorporates the second order with respect to the spatial variables:

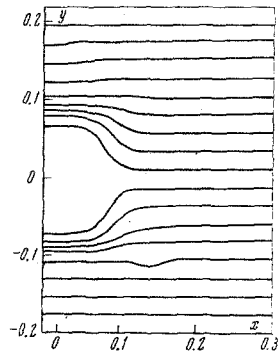


Fig. 1

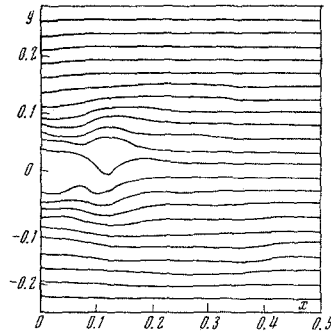


Fig. 2

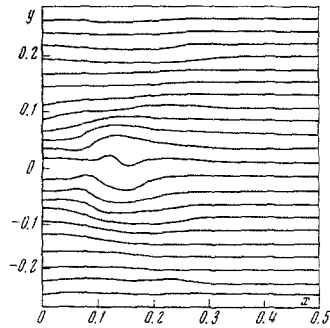


Fig. 3

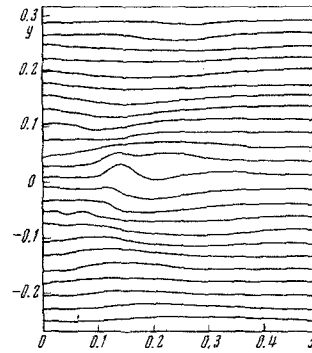


Fig. 4

$$\begin{aligned} \frac{z_{i,j}^{n,k+1/2} - z_{i,j}^{n,k}}{\tau} &= h^{-2} (\rho_{i+1/2,j}^n z_{i+1,j}^{n,k+1/2} - 2\rho_{i,j}^n z_{i,j}^{n,k+1/2} + \rho_{i-1/2,j}^n z_{i-1,j}^{n,k+1/2}) \\ &\quad + h^{-2} (\rho_{i,j+1/2}^n z_{i,j+1}^{n,k} - 2\rho_{i,j}^n z_{i,j}^{n,k} + \rho_{i,j-1/2}^n z_{i,j-1}^{n,k}) - f_{i,j} \\ \frac{z_{i,j}^{n,k+1} - z_{i,j}^{n,k+1/2}}{\tau} &= h^{-2} [\rho_{i,j+1/2}^n (z_{i,j+1}^{n,k+1} - z_{i,j+1}^{n,k}) - \\ &\quad - 2\rho_{i,j}^n (z_{i,j}^{n,k+1} - z_{i,j}^{n,k}) + \rho_{i,j-1/2}^n (z_{i,j-1}^{n,k+1} - z_{i,j-1}^{n,k})] \end{aligned} \quad (4.1)$$

Here  $k$  is the number of interaction,  $\tau$  is the iteration parameter. The iterations end when the difference  $|z_{i,j}^{n,k+1} - z_{i,j}^{n,k}|$  becomes smaller than a certain number  $\varepsilon$  for all the internal points of the mesh. For the quantity  $z_{i,j}^{n,k+1/2}$  homogeneous boundary conditions are applied. It is well known [9] that the scheme of (4.1) has the property of a complete approximation, and in the case of the problem under consideration, is absolutely convergent. Realization of the scheme of (4.1) is effected by the span method.

The determination of

$$\psi_{i,j}^{n+1} = \psi_{i,j}^n + \Delta t z_{i,j}^n + O(\Delta t^2)$$

causes no difficulties; harmonic analysis of this relationship indicates the stability of this stage of the algorithm underlying the solution of the problem, at least for a sufficiently small  $\Delta t$ .

In order to solve the second equation of system (3.1) we use the following scheme of splitting [9]:

$$\frac{p_{i,j}^{n+1/2} - p_{i,j}^n}{\Delta t} = \Lambda_1 \rho_{i,j}^{n+1/2}, \quad \frac{p_{i,j}^{n+1} - p_{i,j}^{n+1/2}}{\Delta t} = \Lambda_2 \rho_{i,j}^{n+1} \quad (4.2)$$

where  $\Lambda_1 \rho$ ,  $\Lambda_2 \rho$  are the difference analogs of the expressions  $(\psi_y \rho)_x + A \rho_{xx}$ ,  $(\psi_x \rho_y)_y + A \rho_{yy}$ , respectively. The splitting scheme (4.2) approximates the second equation of system (3.1) of order  $O(\Delta t + h^2)$  and is absolutely stable.

TABLE 1

$s_1 t$	Results of [2]	Calculation	Percentage difference
0.7703	1.219	1.167	4.27
1.5406	1.463	1.500	2.53
2.3109	1.717	1.667	2.91
3.0812	1.978	2.000	1.11
3.8515	2.162	2.167	0.23
4.6218	2.390	2.500	4.60
5.3921	2.602	2.833	8.88

The boundary conditions for the value of a fractional step are analogous to the corresponding conditions for a complete step. The scheme of (4.2) is realized by the span technique. In order to estimate the accuracy and stability of the algorithm, a numerical study of the behavior of the total energy of the mixture was carried out. The calculations showed that the maximum deviation of the energy from its initial value never exceeded 0.19%.

5. Using the proposed algorithm, we calculated several versions of the problem in hand. In realizing the iterative scheme, the number  $\varepsilon$  was taken within the range 0.1–0.3%  $\max_{i,j} |z_{i,j}^{n,k}|$ . In order to achieve this accuracy,

it was sufficient to make six to ten iterations at each step in time. In order to verify the properties of the mathematical model used for describing the problem, and also the properties of the numerical algorithm underlying the solution of the problem, we made a direct comparison with existing experimental data [2]. It follows from the experimental data of [2] that the horizontal dimension of the spot, referred to the initial value, behaves as the following function:

$$1 + 0.29 (s_1 t)^{1.08} \text{ for } 0 < s_1 t < 2.75$$

$$1.03 (s_1 t)^{0.55} \text{ for } 2.75 \leq s_1 t \leq 25$$

where  $s_1 = (ag/\rho_0)^{1/2}$  is the Wessel–Brandt frequency,  $t$  is the time in sec. By way of the initial horizontal dimension of the spot, the diameter of the cylindrical cavity in which the liquid had a constant density  $\rho_0$  at the initial instant of time was taken in [2]. In the present case this will be the quantity  $l_x$ . We calculated a version of the problem having the following values of the parameters:

$$L_y / L = 0.718, \quad l_x / L = 0.154, \quad l_y / L = 0.180$$

$$A = 0.00014, \quad B = 1.336, \quad \Delta t = 0.0154, \quad \varepsilon = 0.0000008$$

The number of points in the mesh was 40·57. At the initial instant of time the density in the spot is constant; at succeeding instants of time we shall regard the "spot" as meaning a set of mesh points such that the density in these differs by not more than 0.4% from the original, the point  $(i, j)$  belonging to the spot if at least one of the mesh points  $(i, j+1)$ ,  $(i, j-1)$  together with the point  $(i, j)$  satisfies the condition just formulated.

The results of our comparison between the experimental and calculated data are presented in Table 1.

The results of the calculations were extracted every 50 time steps in the form of blocks of values of the functions  $\rho$ ,  $\psi$  at the mesh points and a plot of the lines of equal density at each specific instant of time. Figures 1–4 illustrates the lines of equal density (with an interval of 0.011) at corresponding instants of time; this enables us to follow the initial flattening (collapse) of the spot and the development and propagation of the internal waves. Figures 1–4 relate to the values  $t=0.7703$ , 2.3109, 3.8515, 5.3921, respectively. The results of the calculations agree qualitatively with the results obtained in [2, 3].

#### LITERATURE CITED

1. A. H. Schooley and R. W. Stewart, "Experiment with a self-propelled body submerged in a fluid with a vertical density gradient," *J. Fluid Mech.*, **15**, No. 1, 83–96 (1963).
2. J. Wu, "Mixed region collapse with internal wave generation in density-stratified medium," *J. Fluid Mech.*, **35**, No. 3, 531–544 (1969).
3. R. W. Wessel, "Numerical study of the collapse of a perturbation in an infinite density-stratified fluid," *Phys. Fluids*, **12**, No. 12, Pt. 2, 171–176 (1969).
4. O. Phillips, *Dynamics of the Upper Ocean*, Cambridge Univ. Press (1966).
5. H. Padmanabhan, W. F. Ames, J. F. Kennedy, and Tin-Kan Hung, "A numerical investigation of wake deformation in density-stratified fluids," *J. Engng. Math.*, **4**, No. 3, 229–241 (1970).
6. C. C. Mei, "Collapse of a homogeneous fluid mass in a stratified fluid," *Proc. Twelfth Internat. Congress Appl. Mech.*, Stanford Univ. (1968).
7. A. H. Schooley and B. A. Hughes, "An experimental and theoretical study of internal waves generated by the collapse of two-dimensional mixed region in a density gradient," *J. Fluid. Mech.*, **51**, No. 1, 159–175 (1972).

8. R. I. Hartmann and H. W. Lewis, "Wake collapse in a stratified fluid: linear treatment," *J. Fluid Mech.*, 51, No. 3, 613-618 (1972).
9. N. N. Yanenko, *Method of Fractional Steps in the Solution of Multidimensional Problems of Mathematical Physics* [in Russian], Nauka, Novosibirsk (1967).
10. L. G. Loitsyanskii, *Mechanics of Liquid and Gas* [in Russian], Nauka, Moscow (1970).
11. A. A. Samarskii, *Introduction to the Theory of Difference Schemes* [in Russian], Nauka, Moscow (1971).